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Analytical Expression for the Decay History of an Atmospheric Turbule

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Abstract

This investigation was undertaken as a part of a larger effort to determine the implications of using coherent turbulent structures or turbules to calculate acoustic scattering from turbulent media, a process that has been suggested as a possible non-line-of-sight means of detecting enemy assets on the battlefield. I present an analytical solution to a correlate of the incompressible Navier-Stokes equation. The correlate equation is for the enstrophy, the square of the vorticity. The solution is an expression for the time history of the enstrophy for a particular choice of the initial velocity distribution.

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1. Introduction

This investigation was undertaken as part of a larger effort to determine the implications of using coherent turbulent structures to calculate acoustic scattering from turbulent media, a process that has been suggested as a possible non-line-of-sight means for detecting enemy assets on the battlefield. I have published a number of papers on the scattering of acoustic signals from atmospheric turbulence on the battlefield, as well as a new way of modeling turbulence [1]. My new modeling method is called the Turbule Ensemble Model (TEM). The TEM method consists of representing the turbulence region as a collection of *turbules* (eddies) of prescribed sizes and random locations. A turbule is an isolated inhomogeneity in either the temperature or the velocity. Methods for determining the parameter values required to complete the calculation of the signal scattered into a shadow zone have been reported elsewhere [1].

As methods of determining the parameter values have improved, reexamination of assumptions in light of theory/experiment comparisons has proceeded apace. One of these assumptions is that Taylor's frozen turbulence hypothesis holds. Previous work [1] indicates that an interplay of the scattering geometry and the location of the shadow zone boundary determines the turbule sizes that must be considered in any particular scenario. For example, a high boundary reduces the contribution of large turbules because the scattering pattern of large turbules is concentrated in the forward direction. This means that the scattered signal passes above the detector, which is usually located near the ground. The persistence of smaller turbules is shorter than that of larger turbules, as can be seen from dimensional analysis of the relevant flow parameters [2]. As is characteristic of dimensional analysis results, the constant multiplier to be used with the rate so determined is unknown. Since the rate constant is usually included in an exponent, the effect of not knowing the multiplier more precisely can give improper indications as to the source of any discrepancy between predicted and measured results. As the scenario model is honed ever nearer to the correct representation of the experiment, improved accuracy in parameter estimation becomes more important.

The original purpose of the work reported here was to come up with a better estimate of the time interval over which Taylor's frozen turbulence hypothesis may be relied upon. As detailed in section 6, this objective was not reached, in a practical sense, because the analysis is for an isolated velocity distribution. The mathematical development of the investigation is reported here because it may be useful in research on decay times and perturbation effects as the Reynolds number transitions from the laminar to the turbulent regime. A short synopsis of the mathematical development follows.

Since the interest at this time is on the evolution of a velocity turbule, the Navier-Stokes (N-S) equation is the logical place to begin such an investigation, because the equation relates the rate of change of fluid velocity to other flow quantities. The convective acceleration portion of the N-S

equation (the velocity equation) can be transformed by a vector identity into the cross product of the velocity and the vorticity, plus the gradient of the square of the velocity. When the curl of the resulting equation is performed (assuming constant density and viscosity), all gradient terms drop out, leaving a second equation (the vorticity equation), which comprises two terms in the vorticity and one other term. The latter term is the curl of the velocity-vorticity cross product. For a particular initial velocity distribution, namely the cross product of the angular velocity and the position vector multiplied by a Gaussian envelop function, the latter term turns out to be parallel to the velocity. Since the vorticity associated with this velocity distribution is perpendicular to the velocity, the dot product of the vorticity with the latter term is initially zero. Thus, a third equation (the enstrophy equation) formed by the dot product of the vorticity and the vorticity equation has the interesting property that the cross-product term is initially zero. The term enstrophy is defined elsewhere [3] as the time average of the squared vorticity. Enstrophy as used here refers to the square of the vorticity. In a further step in the analysis, a trial time-dependent enstrophy distribution expression was derived by solving the enstrophy equation, assuming the cross product term is zero. A surprising result is that the time-dependent vorticity and velocity distributions inferred from the time-dependent enstrophy distribution produced a cross-product term in the enstrophy equation that was identically zero for all time. These velocity/vorticity expressions are not solutions to the velocity equation or to the vorticity equation. The expression for the time history of the enstrophy is unusual to the extent that the standard definition of the time constant is no longer appropriate.

The organization of the remainder of the report is as follows: In section 2, transformation of the N-S velocity equation into an equation for the vorticity is outlined. Section 3 contains an analysis of the characteristics of the N-S convective derivative term (as transformed into vorticity) for the particular initial velocity distribution that I have chosen. I show that the dot product of the convective derivative term and the vorticity is zero and, therefore, the dot product of the vorticity with the vorticity equation eliminates this term from immediate consideration. Such a dot product without the convective derivative term produces a differential equation in the enstrophy only. In section 4, Fourier transform (FT) and Laplace transform (LT) theories are used to solve for the time-dependent enstrophy field. In section 5, time-dependent velocity and vorticity fields are inferred from the enstrophy field and inserted into the original enstrophy equation. This step confirms that the inferred time-dependent fields, like the initial fields, produce a zero convective derivative term, and that time-derivative and Laplacian terms counterpoise each other so that the inferred fields are a consistent solution to the enstrophy equation. Section 6 contains calculated results and recommendations for future work. Section 7 is a brief summary. Appendix A is a short review of applicable FT theory and notation. Appendix B contains details of the equation-solution procedure. Appendix C contains details of the confirmation process. Appendix D contains proof that the time-dependent velocity field inferred in section 5 is not a solution to the N-S equation.

2. The Navier-Stokes Equation for Vorticity

The N-S equation is given [4] as the following:

$$\frac{\partial u}{\partial \tau} + (u \cdot \nabla)u = -\left(\frac{1}{\rho}\right)\nabla p + \left(\frac{\eta}{\rho}\right)\nabla^2 u + \left(\frac{\zeta}{\rho} + \frac{\eta}{3\rho}\right)\nabla(\nabla \cdot u) \ . \tag{1}$$

Following the reference, this equation for the velocity u and the pressure p may be converted into an expression in which the gradient terms are absent. Using the following standard vector formulas [5] given in the first line of equation (2), the second term of equation (1) may be transformed as

$$\begin{aligned} (1/2)\nabla u^2 &= u \times (\nabla \times u) + (u \cdot \nabla)u \\ \frac{\partial u}{\partial \tau} &= -\left(\frac{1}{\rho}\right)\nabla p - (1/2)\nabla u^2 + u \times (\nabla \times u) + \\ &\quad \left(\frac{\eta}{\rho}\right)\nabla^2 u + \left(\frac{\zeta}{\rho} + \frac{\eta}{3\rho}\right)\nabla(\nabla \cdot u) \ . \end{aligned} \tag{2}$$

If the density ρ and the viscosity coefficients η and ζ are constant, taking the curl of both sides results in the following relation:

$$\begin{aligned} \nabla \times (\nabla U) &= 0; \quad \text{if } U \text{ is a scalar function} \\ d &= \nabla \times u \\ \frac{\partial d}{\partial \tau} &= \nabla \times (u \times d) + \nu \nabla^2 d \, . \end{aligned} \tag{3}$$

The first expression in equation (3) is a standard vector formula [5]. The second expression defines the vorticity d. The kinematic viscosity coefficient is $\nu = \eta/\rho$, where η is the dynamic viscosity coefficient. In equation (3) the functions (u, d) are functions of the space coordinates y_j, $(j = 1, 2, 3)$, whose units are typically the meter, and the time τ, whose unit is typically the second. As noted in the introduction, the interest of this report is in the dynamics of an atmospheric velocity turbule. A velocity turbule has a characteristic linear dimension a and a characteristic angular velocity magnitude Ω_0. These parameters are conveniently used to convert equation (3) to the unitless coordinates x, the unitless time t, the unitless velocity v, and the unitless vorticity c, according to the relations in the following equation:

$$\begin{aligned} &\frac{\partial c}{\partial t} = \nabla_x \times (v \times c) + K \nabla_x^2 c \ , \\ &\text{where } x_j = y_j / a; \quad t = \Omega_0 \tau; \quad v = u/(\Omega_0 a); \quad c = d/\Omega_0; \\ &\nabla_x = a \nabla_y; \quad K = \nu/(a^2 \Omega_0); \quad \partial_t = \frac{\partial}{\partial t}; \quad \partial_i = \frac{\partial}{\partial x_i} . \end{aligned} \tag{4}$$

The notation used above is summarized as $i = 1, 2, 3$ = subscript index (repeated subscripts imply summation), x_i = space variables, $\hat{x}_i$ = space unit vectors, and x = space vector = $\hat{x}_i x_i$.

The parameter K is the reciprocal of the typical Reynolds number. The symbol K is used rather than the symbol R_e^{-1} for writing economy. The subscript x will be dropped from the ∇ operator in subsequent developments and, where appropriate, the shorthand notation for partial derivatives indicated in the last line of equation (4) will be used for the same reason.

3. The Nature of the N-S Equation's Second Term

The second term referred to in the N-S equation is the curl of the cross product of v and c in equation (4) for which the symbol $s(x,t)$ is reserved. The class of solutions sought is for rotationally symmetric velocity distributions. A member of this class is the distribution defined in the equation

$$u(y) = [\Omega \times y]\exp[-(y_i/a)^2] \ . \tag{5}$$

In equation (5), Ω is an angular velocity vector and a is the characteristic turbule size. No loss of generality is suffered if $u(x)$ is further particularized by choosing Ω to be aligned with the y_3 axis as follows:

$$\begin{aligned} \Omega &= \hat{y}_3\Omega_0 \\ u_0(y) &= \Omega_0[-\hat{y}_1 y_2 + \hat{y}_2 y_1]\exp[-(y_i/a)^2] . \end{aligned} \tag{6}$$

Converting to unitless quantities using equation (4) results in the following velocity distribution:

$$\begin{aligned} x &= y/a; \qquad v_0 = u_0/(a\Omega_0) \\ v_0(x) &= (-\hat{x}_1 x_2 + \hat{x}_2 x_1)\exp[-(x_i)^2] . \end{aligned} \tag{7}$$

The corresponding vorticity function follows from equations (3) and (4):

$$c_0(x) = 2[\hat{x}_1 x_1 x_3 + \hat{x}_2 x_2 x_3 + \hat{x}_3(1 - x_1^2 - x_2^2)]\exp[-(x_i)^2] \ . \tag{8}$$

The dot product of the velocity and vorticity is seen to be zero, so the two are perpendicular.

The crossproduct of the velocity and the vorticity is

$$v_0 \times c_0 = 2[\hat{x}_1 x_1(1 - x_1^2 - x_2^2) + \hat{x}_2 x_2(1 - x_1^2 - x_2^2) - \hat{x}_3(x_1^2 + x_2^2)x_3]\exp[-2(x_i)^2] \ . \tag{9}$$

Take the curl of equation (9) to obtain the second term, which is

$$s_0 = 4[\hat{x}_1 x_2 - \hat{x}_2 x_1]x_3\exp[-2(x_i)^2] \ . \tag{10}$$

The second term s_0 is seen to be oppositely directed to v_0. This means that the dot product of the vorticity and s_0 is zero. The equation to be solved is, therefore, the following:

$$\frac{1}{2}\partial_t c^2 = c \cdot \nabla \times (v \times c) + K c \cdot \nabla^2 c \ . \tag{11}$$

This equation is referred to as the transport equation for instantaneous enstrophy [6] and was used for a different purpose there. When (v_0, c_0) are substituted in equation (4), the second term is eliminated. The procedure

to be followed, then, is to solve equation (11) without the v cross c term for the initial vorticity distribution of equation (8), and then substitute the result in equation (11), including the v cross c term, to confirm that the answer leads to a zero second term for all time, and that the other terms combine to make equation (11) true for all time.

4. Transformation of the N-S Equation

FT theory will be used in section 4 to solve equation (11), less the v cross c term, acknowledging in the process that this term is the very essence of turbulence. The applicable theory is outlined in appendix A, along with the notation employed, and the definition of the FT of a function.

The convention used for field functions is: lower-cased letters are in the space domain and upper-cased letters are in the FT or wavenumber domain. Additionally, (v, V) = velocity field function (vectors), and (c, C) = vorticity field function (vectors).

Additional notation is q_i = wavenumber variables, $\hat{q}_i$ = wavenumber unit vectors, and q = wavenumber vector = $\hat{q}_i q_i$.

A general property of the velocity distributions associated with incompressible flow is that the divergence is zero, as in

$$\nabla \cdot v = \partial_i v_i = 0 \ . \tag{12}$$

It is easy to verify by direct application of the divergence operator that the divergence of the vorticity is also zero, as in

$$\nabla \cdot c = \partial_i c_i = 0 \ . \tag{13}$$

The FT of equations (12) and (13) give the following result:

$$\begin{aligned} q_i V_i &= 0 \\ q_i C_i &= 0 . \end{aligned} \tag{14}$$

The FT of the velocity, vorticity, and second-term functions from equations (7), (8), and (10) are as follows:

$$\begin{aligned} V_0(q) &= j\pi^{3/2}[-\hat{q}_1 q_2 + \hat{q}_2 q_1]\exp[-q_i^2/4]/2 \\ C_0(q) &= \pi^{3/2}\{-\hat{q}_1 q_1 q_3 - \hat{q}_2 q_2 q_3 + \hat{q}_3[q_1^2 + q_2^2]\}\exp[-q_i^2/4]/2 \\ S_0(q) &= (\pi^3/2)^{1/2}\{-\hat{q}_1 q_2 + \hat{q}_2 q_1\} q_3 \exp[-q_i^2/8]/8 . \end{aligned} \tag{15}$$

In the FT domain, the equations giving the vorticity in terms of the velocity are as follows:

$$\begin{aligned} C_1 &= j(q_2 V_3 - q_3 V_2) \\ C_2 &= j(q_3 V_1 - q_1 V_3) \\ C_3 &= j(q_1 V_2 - q_2 V_1) . \end{aligned} \tag{16}$$

A useful relation is the sum of the above three components:

$$C_1 + C_2 + C_3 = j[(q_3 - q_2)V_1 + (q_1 - q_3)V_2 + (q_2 - q_1)V_3] \; . \qquad (17)$$

Combining equations (14), (16), and (17) gives the following results for the V_i in terms of the C_i:

$$\begin{aligned} V_1 &= -j\left[\frac{(q_2C_3 - q_3C_2)}{q_i^2}\right] \\ V_2 &= -j\left[\frac{(q_3C_1 - q_1C_3)}{q_i^2}\right] \\ V_3 &= -j\left[\frac{(q_1C_2 - q_2C_1)}{q_i^2}\right]. \end{aligned} \qquad (18)$$

The results of equation (18) allow substitution of vorticity components for velocity components.

Rewriting equation (11) using the tensor notation of the fourth line in equation (4) and dropping the cross-product term results in

$$\begin{aligned} c_j(x,t)\partial_t c_j(x,t) &= K c_j(x,t)\partial_i^2 c_j(x,t) \\ \partial_t[c_j(x,t)]^2 &= 2K c_j(x,t)\partial_i^2 c_j(x,t) \, . \end{aligned} \qquad (19)$$

Also indicated in equation (19) is the fact that the vorticity is a function of time. A useful modification of equation (19) is obtained by consideration of the second partial of the square of some function, such as $f(x)$, as in

$$\begin{aligned} \partial_x^2[f(x)]^2 &= \partial_x\{\partial_x[f(x)]^2\} = \partial_x\{2f(x)\partial_x f(x)\} = 2f(x)\partial_x^2 f(x) + 2[\partial_x f(x)]^2 \\ f(x)\partial_x^2 f(x) &= (1/2)\partial_x^2[f(x)]^2 - [\partial_x f(x)]^2 \, . \end{aligned} \qquad (20)$$

The last line of equation (19) is rewritten for the first component using the last line of equation (20), as in

$$\partial_t[c_1(x,t)]^2 = K\{\partial_i^2[c_1(x,t)]^2 - 2[\partial_i c_1(x,t)]^2\} \; . \qquad (21)$$

Similar expressions may be written for the other components of $c(x,t)$. The next step is to obtain the FT of the above expression through the equation

$$\begin{aligned} \partial_t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (dq')^3 C_1(q',t)C_1(q - q',t) &= -Kq_i^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (dq')^3 c_1(q',t)c_1(q - q',t) + \\ & 2K \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (dq')^3[(q_i')(q_i - q_i')]C_1(q',t)C_1(q - q',t) \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (dq')^3[\partial_t + Kq_i^2 - 2K(q_i')(q_i - q_{i'})]C_1(q',t)C_1(q - q',t) &= 0 \, . \end{aligned} \qquad (22)$$

Equation (22) is solved in appendix B. The symbol $e(x,t)$ will be used for the enstrophy in

$$
\begin{aligned}
e(x,t) &= [c_j(x,t)]^2 \\
\mathcal{F}\{e(x,t)\} &= E(q,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 C_j(q',t)C_j(q-q',t) .
\end{aligned}
\tag{23}
$$

From appendix B, the IFT of the last expression in equation (23) may now be written as

$$
e(x,t) = \left(\frac{4\exp[-2x_i^2/(4L)]}{(4L)^7}\right)[(x_1x_3)^2 + (x_2x_3)^2 + (4L - x_1^2 - x_2^2)^2] . \tag{24}
$$

Equation (24) is the most significant and novel result in this report. Notation has been simplified by the substitution $L = 1/4 + Kt$. As an elementary check of mathematical manipulations, it is easy to see from the above that the initial ($t = 0$) enstrophy distribution is indeed the square of $c_0(x)$ from equation (8).

5. Confirmation of Results

The purpose of this section is to confirm that the results derived in section 4 are, indeed, a solution of the transport equation for instantaneous enstrophy (eq (11)). The original differential equation (eq (4)) is repeated here:

$$
\partial_t c = \nabla\times(v\times c) + K\nabla^2 c . \tag{25}
$$

Equation (25) is actually three equations, wherein the operations are on the components of the vorticity. Again using the symbol s for the second term, the three equations can be represented as one equation in tensor notation, using the coordinate unit vectors

$$
\partial_t c_j\hat{x}_j = s_j\hat{x}_j + K(\partial_i)^2 c_j\hat{x}_j . \tag{26}
$$

The dot product was applied to this vector equation, resulting in equation (11), which is reproduced here:

$$
c_j\partial_t c_j = c_j s_j + Kc_j(\partial_i)^2 c_j . \tag{27}
$$

It is also necessary to verify that our solution satisfies the divergence equations (12) and (13). To perform this verification, the solution will consist of trial expressions, which will be substituted into equation (12), (13), and (27). Trial functions will be designated by bold letters.

The trial vorticity function is contained in the components of equation (24), or

$$
\begin{aligned}
\mathbf{c} &= \left(\frac{2\exp[-x_i^2/(4L)]}{(4L)^{7/2}}\right)[\hat{x}_1x_1x_3 + \hat{x}_2x_2x_3 + \hat{x}_3(4L - x_1^2 - x_2^2)] \\
L &= 1/4 + Kt .
\end{aligned}
\tag{28}
$$

Note that the divergence of the above is zero as can be seen by summing the three components in equation (29):

$$\begin{aligned}
\partial_1 c_1 &= \left(\frac{2\exp[-x_i^2/(4L)]}{(4L)^{7/2}}\right)[1 - 2x_1^2/(4L)]x_3 \\
\partial_2 c_2 &= \left(\frac{2\exp[-x_i^2/(4L)]}{(4L)^{7/2}}\right)[1 - 2x_2^2/(4L)]x_3 \\
\partial_3 c_3 &= \left(\frac{2\exp[-x_i^2/(4L)]}{(4L)^{7/2}}\right)[-2(4L - x_1^2 - x_2^2)x_3/(4L)] \ .
\end{aligned} \tag{29}$$

There is a direct link between the trial vorticity (eq (28)) and the trial velocity: find the FT of **c**; apply equation (18) to find the FT of **v**; and find the inverse Fourier transform (IFT) of the result. This has not yet been done. An alternate route is to propose a functional form in analogy with equations (7) and (28), including a multiplicative factor; find the curl of that form; and derive the specific form of the factor to make the result equivalent to equation (28). The proposed form is in equation (30), where A is the *ab initio* unknown factor, as in

$$\begin{aligned}
\mathbf{v} &= A(-\hat{x}_1 x_2 + \hat{x}_2 x_1)\exp[-x_i^2/(4L)] \\
\nabla\times\mathbf{v} &= [2A/(4L)][\hat{x}_1 x_1 x_3 + \hat{x}_2 x_2 x_3 + \hat{x}_3(4L - x_1^2 - x_2^2)]\exp[-x_i^2/(4L)] \\
2A/(4L) &= 2/(4L)^{7/2}; \quad A = 1/(4L)^{5/2} \\
\mathbf{v} &= [1/(4L)^{5/2}](-\hat{x}_1 x_2 + \hat{x}_2 x_1)\exp[-x_i^2/(4L)] \ .
\end{aligned} \tag{30}$$

It is obvious that the divergence of **v** is zero because of the opposite sign of the two terms, and the fact that the derivatives of the exponential factor will make their magnitudes the same.

To record the verification process cogently, set up the Λ_c function, comprising all three separately identified terms of equation (27), shifted to the left-hand side as in

$$\begin{aligned}
\Lambda_c &= \boldsymbol{\alpha}_c - \boldsymbol{\beta}_c - K\boldsymbol{\varphi}_c \\
\boldsymbol{\alpha}_c &= \mathbf{c}_j \partial_t \mathbf{c}_j \\
\boldsymbol{\beta}_c &= \mathbf{c}_j \mathbf{s}_j \\
\boldsymbol{\varphi}_c &= \mathbf{c}_j (\partial_i)^2 \mathbf{c}_j \ .
\end{aligned} \tag{31}$$

Showing Λ_c is zero will prove that the (**c**, **v**) functions from equations (28) and (30) are a consistent solution set for equation (27). Expansion and simplification of functions ($\boldsymbol{\alpha}_c$, $\boldsymbol{\beta}_c$, $\boldsymbol{\varphi}_c$) are done in appendix C. The results are that $\boldsymbol{\beta}_c = 0$ and $\boldsymbol{\alpha}_c = K\,\boldsymbol{\varphi}_c$. Therefore, the conclusion is that (**c**, **v**) are a consistent solution set for equation (27).

6. Discussion of Results

Displaying the nature of the enstrophy field defined by equation (24) is somewhat of a challenge because of the violent variability with respect to time. The size/time scales are incorporated in the constant K, so that any curves in terms of the variables $(x, t/K)$ are universal. If t is $1/(4K)$, the enstrophy at the origin $(x = 0)$ is already 1/32 of the value for $t = 0$. Ordinarily, a time constant is defined when some energy like quantity has declined to one-half of its starting value. This would translate into the $t = 1/(4K)$ if the numerator/denominator combination was L^{-1}. The fact that the numerator/denominator combination changes when $x \neq 0$ further complicates the search for a simple describer for the time history. If the size is defined to be the radius of the circle in the (x_1, x_2) plane, where the exponent of envelope function is -2, then the size $a(t) = (4L)^{1/2}$. The meaning of this is that the turbule gets larger with time, a rather novel concept. This apparent increase in size is masked by the 5th power fall off of the central maximum of the distribution. From an acoustic scattering point of view, the scattering cross section declines with time, but the differential scattering cross section becomes more peaked in the forward direction. The enstrophy is zero at certain points in the $(x_1 - x_2)$ plane reflecting the fact that the vorticity changes sign for this particular starting velocity distribution as the point of interest recedes from the axis.

To aid intuitive understanding, the dimensionless fields and variables will be discarded and the pertinent equations will be as follows:

$$\begin{aligned}
d(y,\tau) &= \left(\frac{2\Omega_0 a^5 \exp[-y_i^2/(a^2 + 4\nu\tau)]}{(a^2 + 4\nu\tau)^{7/2}}\right)[\hat{y}_1 y_1 y_3 + \hat{y}_2 y_2 y_3 + \\
&\qquad \hat{y}_3(a^2 + 4\nu\tau - y_1^2 - y_2^2)] \\
u(y,\tau) &= \left(\frac{\Omega_0 a^5 \exp[-y_i^2/(a^2 + 4\nu\tau)]}{(a^2 + 4\nu\tau)^{5/2}}\right)[-\hat{y}_1 y_2 + \hat{y}_2 y_1] \\
e(y,\tau) &= \left(\frac{4\Omega_0^2 a^{10} \exp[-2y_i^2/(a^2 + 4\nu\tau)]}{(a^2 + 4\nu\tau)^7}\right)[(y_1 y_3)^2 + (y_2 y_3)^2 + \\
&\qquad (a^2 + 4\nu\tau - y_1^2 - y_2^2)^2] \\
w(y,\tau) &= \left(\frac{\rho\Omega_0^2 a^{10} \exp[-2y_i^2/(a^2 + 4\nu\tau)]}{2(a^2 + 4\nu\tau)^5}\right)[y_1^2 + y_2^2].
\end{aligned} \tag{32}$$

Perhaps the best way to gain some numerical idea as to the variations of the turbule enstrophy is to consider the total of this quantity. As a matter of convenience this is done in the following equation, where the volume integral of the enstrophy is calculated as:

$$\mathcal{E}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dy)^3 e(y,\tau) = \left(\frac{5\pi^{3/2}\Omega_0^2 a^{10}}{4\sqrt{2}(a^2 + 4\nu\tau)^{7/2}}\right). \tag{33}$$

The new function $w(y,\tau)$ in equation (32) is the kinetic energy concentration. The total kinetic energy is calculated from the following expression:

$$W(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dy)^3 w(y,\tau) = \left(\frac{\pi^{3/2}\Omega_0^2 a^{10}\rho}{8\sqrt{2}(a^2 + 4\nu\tau)^{5/2}}\right)$$
$$\frac{W(\tau)}{W(0)} = (4\mathrm{L})^{-5/2}. \tag{34}$$

From the last expression in equation (34), let $t_{1/2}$ be that time for which the ratio is 1/2 (a condition that occurs when $K t_{1/2} = (2^{2/5} - 1)/4 = 0.0798770$). The intent is now to investigate the state for seven different turbule sizes. The relevant data is collected in table 1.

The sizes are from the ensemble reported previously [1]. In this ensemble, the characteristic size of the largest-sized turbule was $a_1 = 10$ m. The maximum velocity in the largest turbule was approximately 1/100 of the speed of sound or $v_1 = 3.44\ \mathrm{m \cdot s^{-1}}$. The turbule size ratio was $a_n/a_{n+1} = 0.827787$. For a velocity structure constant of 0.1111 $\mathrm{m^{4/3} \cdot s^{-2}}$, the turbule spacing parameter was $\ell = 8$. From equation (6), the maximum velocity occurs for $y_3 = 0$ on a circle with a radius of $\rho_m = 2^{-1/2} a$. The kinematic viscosity coefficient is taken to be $0.15 \times 10^{-4}\ \mathrm{m^2 \cdot s^{-1}}$. The starting maximum velocity of the different-sized turbules is assumed to vary according to the (1/3) power rule applied to the size ratio. The critical Reynolds number for boundary layer flow with zero pressure gradient is given [7] as about 600, indicating the value when the flow becomes unstable. The sizes in the table cover a Reynolds number range spanning this value.

Other valuable information is provided in table 2, including data on total kinetic energy and the pressure differential Δp at the origin, which is calculated from the following relationship to the velocity in the $y_1 - y_2$ plane:

$$r dp = \rho(u^2/r) r dr \qquad r = \sqrt{y_1^2 + y_2^2}$$
$$\Delta p(\tau) = 2\int_0^{\infty} dr\, w(r,0,\tau)/r = \left(\frac{\Omega_0^2 a^{10}\rho}{4(a^2 + 4\nu\tau)^4}\right). \tag{35}$$

Table 1. Turbule decay data.

Characteristic size, a (m)	Maximum velocity, v_m ($\mathrm{m \cdot s^{-1}}$)	Radius for maximum, ρ_m (m)	Angular velocity, Ω_0 ($\mathrm{s^{-1}}$)	Decay constant, K	Relative half-time, $t_{1/2}$	Half-time, $\tau_{1/2}$ (s)	Enstrophy int., $\mathscr{E}(y, \tau_{1/2})$ ($\mathrm{m^3 \cdot s^{-2}}$)
1.04×10	1.62×10	7.32×10^{-1}	1.63×10^{1}	8.58×10^{-7}	9.30×10^{4}	5.71×10^{3}	3.10×10^{2}
3.33×10^{-1}	1.11×10	2.36×10^{-1}	3.47×10^{1}	3.89×10^{-6}	2.05×10^{4}	5.91×10^{2}	6.19×10^{1}
1.07×10^{-1}	7.58×10^{-1}	7.58×10^{-2}	7.40×10^{1}	1.77×10^{-5}	4.52×10^{3}	6.12×10^{1}	9.35×10
3.45×10^{-2}	5.20×10^{-1}	2.44×10^{-2}	1.58×10^{2}	8.01×10^{-5}	9.97×10^{2}	6.33×10	1.41×10
1.11×10^{-2}	3.56×10^{-1}	7.84×10^{-3}	3.35×10^{2}	3.63×10^{-4}	2.20×10^{2}	6.55×10^{-1}	1.79×10^{-1}
3.57×10^{-3}	2.44×10^{-1}	2.52×10^{-3}	7.14×10^{2}	1.65×10^{-3}	4.85×10^{1}	6.78×10^{-2}	1.79×10^{-1}
1.15×10^{-3}	1.67×10^{-1}	8.12×10^{-4}	1.52×10^{3}	7.48×10^{-3}	1.07×10^{1}	7.02×10^{-3}	1.79×10^{-1}

Note: $\nu = 1.50 \times 10^{-5}\ m^2 \cdot s^{-1}$ *and* $K t_{1/2} = 0.079877$.

The maximum pressure differential (in the largest turbule) is about 86 Pa, compared to the nominal atmospheric pressure of 101,000 Pa. These pressure data confirm that the constant density assumption is, essentially, valid.

A comparison between the decay time of this theory and the decay time predicted by Kolmogorov's theory is instructive. The relation from which the latter time may be calculated is as follows:

$$\tau_K \sim (a^2/\varepsilon)^{1/3} \ . \tag{36}$$

In equation (36), ε is the energy-decay rate per unit mass. A typical value [8] for it is $\varepsilon = 10^{-3}\ m^2 \cdot s^{-3}$. Table 3 contains the relative comparative data.

From table 3, we see that the comparable times occur when the size is around 1 cm. The theory of this report shows a steeper decline for smaller sizes than Kolmogorov's prediction, suggesting that the present theory might be useful for analyzing the turbule interaction in this region. For example, consider two turbules with initial velocity distributions similar in form to that considered in the table, separated by a suitable distance. The only terms in equation (11) initially present would be the interaction terms, and the decay (or growth) of the interaction could prove to be very interesting.

Table 2. Ancillary turbule data.

Characteristic size, a (m)	Maximum velocity, v_m ($m \cdot s^{-1}$)	Radius for maximum, ρ_m (m)	Angular velocity, Ω_0 (s^{-1})	Kinetic energy, W (J)	Half-kin. energy $W_{1/2}$ (J)	Center pressure Δp (Pa)	Half-center pressure $\Delta p_{1/2}$ (Pa)
1.04×10	1.62×10	7.32×10^{-1}	1.63×10^{1}	1.88×10^{2}	9.41×10^{1}	8.62×10^{1}	2.84×10^{1}
3.33×10^{-1}	1.11×10	2.36×10^{-1}	3.47×10^{1}	2.94×10	1.79×10	4.05×10^{1}	1.33×10^{1}
1.07×10^{-1}	7.58×10^{-1}	7.58×10^{-2}	7.40×10^{1}	4.60×10^{-2}	2.81×10^{-2}	1.90×10^{1}	6.27×10
3.45×10^{-2}	5.20×10^{-1}	2.44×10^{-2}	1.58×10^{2}	7.20×10^{-4}	4.39×10^{-4}	8.92×10	2.94×10
1.11×10^{-2}	3.56×10^{-1}	7.84×10^{-3}	3.35×10^{2}	1.13×10^{-5}	6.86×10^{-6}	4.19×10	1.38×10
3.57×10^{-3}	2.44×10^{-1}	2.52×10^{-3}	7.14×10^{2}	1.76×10^{-7}	1.07×10^{-7}	1.97×10	6.49×10^{-1}
1.15×10^{-3}	1.67×10^{-1}	8.12×10^{-4}	1.52×10^{3}	2.75×10^{-9}	1.68×10^{-9}	9.24×10^{-1}	3.05×10^{-1}

$p_0 = 1.01 \times 10^5$ *Pa* $\rho = 1.21$ *kg* $\cdot m^{-3}$.

Table 3. Decay time comparison with Kolmogorov prediction.

a (m)	1.04×10	3.33×10^{-1}	1.07×10^{-1}	3.45×10^{-2}	1.11×10^{-2}	3.57×10^{-3}	1.15×10^{-3}
τ_K (s)	1.03×10^{1}	4.80×10	2.25×10	1.06×10	4.98×10^{-1}	2.34×10^{-1}	1.10×10^{-1}
$\tau_{1/2}$ (s)	5.71×10^{3}	5.91×10^{2}	6.12×10^{1}	6.33×10	6.55×10^{-1}	6.78×10^{-2}	7.02×10^{-3}

7. Conclusion

The mathematical developments reported in this document contain a valid solution to the enstrophy correlate of the N-S equation, giving the time history of the decay of the enstrophy from a particular choice of the initial velocity distribution of an isolated velocity turbule. There may be other initial velocity distributions that lead to a similar analytical solution.

We found that the time-dependent velocity field derived here is not a solution to equation (2), the N-S equation. Details of this analysis are given in appendix D.

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Appendix A.—Applicable Fourier Transform Theory

Fourier transform (FT) theory was used in section 4 of the main report to solve the equation resulting from forming the dot product of the vorticity and equation (3). The applicable theory is outlined in this appendix, along with the notation employed.

The FT of function $g(y)$ of variable y to variable f is defined as follows:

$$\mathcal{F}\{g(y),f\} = G(f) = \int_{-\infty}^{\infty} dy\, g(y) \exp(jfy) \quad . \qquad \text{(A-1)}$$

The inverse FT (IFT) of function $G(f')$ of variable f' to variable y' is defined as follows:

$$\mathcal{F}^{-1}\{G(f'),y'\} = g(y') = (2\pi)^{-1} \int_{-\infty}^{\infty} df'\, G(f') \exp(-jf'y') \quad . \qquad \text{(A-2)}$$

The Dirac delta function, the symbol for which is $\delta(y)$, and its FT are of considerable importance. $\delta(y)$ is defined by the limiting process recorded below:

$$\begin{aligned} \delta(y) &= \lim_{A \to 0} \left(\frac{1}{A}\right); & -A/2 \le y \le A/2 = 0 \\ &= 0; & y < -A/2 \text{ or } A/2 \le y \quad . \end{aligned} \qquad \text{(A-3)}$$

The integral of $\delta(y)$ is, therefore, unity. The FT of $\delta(y)$ is the following:

$$\begin{aligned} \mathcal{F}\{\delta(y)\} &= \int_{-\infty}^{\infty} dy\, \delta(y) \exp(jfy) = \lim_{A \to 0} \left(\frac{1}{A}\right) \int_{-A/2}^{A/2} dy\, \exp(jfy) = \\ &\lim_{A \to 0} \left[\frac{\exp(jfy)}{jAf}\right]_{-A//2}^{A/2} = \lim_{A \to 0} \left[\frac{2\sin(Af/2)}{Af}\right] = 1 \; . \end{aligned} \qquad \text{(A-4)}$$

The FT of $\delta(y)$ is, therefore, a constant; i.e., it has a uniform spectrum. The IFT of 1, then, recovers the delta function

$$\mathcal{F}^{-1}\{1\} = (2\pi)^{-1} \int_{-\infty}^{\infty} df\, \exp(-jfy') = \delta(y') \quad . \qquad \text{(A-5)}$$

When this integral construction is encountered, substitution of a suitable delta function is appropriate.

The next few expressions define the convolution integral, which is useful when dealing with the product of two functions.

$$f(y) = g(y)h(y)$$

$$\mathcal{F}\{f(y),f\} = F(f) = \int_{-\infty}^{\infty} dy\, g(y)h(y)\exp(jfy) \quad \cdot \qquad \text{(A-6)}$$

In equation (A-6), $f(y)$ is defined as the product of the functions $g(y)$ and $h(y)$. The FT of $f(y)$ is defined the standard way. To develop the convolution integral, $g(y)$ and $h(y)$ are expressed in terms of the IFT of their transforms $G(f)$ and $H(f)$, as in

$$\begin{aligned} F(f) &= \int_{-\infty}^{\infty} dy\,(2\pi)^{-1} \int_{-\infty}^{\infty} df'G(f')\exp(-jf'y)\,(2\pi)^{-1} \int_{-\infty}^{\infty} df''H(f'')\exp(-jf''y)\exp(jfy) \\ &= (2\pi)^{-2} \int_{-\infty}^{\infty} df'G(f') \int_{-\infty}^{\infty} df''H(f'') \int_{-\infty}^{\infty} dy\, \exp[(jy)(f - f' - f'')] \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} df'G(f') \int_{-\infty}^{\infty} df''H(f'')\delta(f - f' - f'') \qquad \text{(A-7)} \\ &= (2\pi)^{-1} \int_{-\infty}^{\infty} df'G(f')H(f - f') = (2\pi)^{-1}\, G(f) * H(f) \ . \end{aligned}$$

The last line of equation (A-7) defines the convolution integral. The rightmost member of the last line shows how the integral is sometimes represented. The convolution integral is unchanged by interchange of the functional dependencies between *G* and *H*.

The three-dimensional FT of function $h(x)$ of vector variable x to vector variable q is defined as

$$\mathcal{F}\{h(x),q\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (dx)^3 h(x)\exp(jq_i x_i) = H(q) \ . \qquad \text{(A-8)}$$

The IFT of function $H(q')$ of vector variable q' to vector variable x' is shown as

$$\mathcal{F}^{-1}\{H(q'),x'\} = (2\pi)^{-3} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (dx')^3 H(q') \exp(-jq_i'x_i') = h(x') \ . \qquad \text{(A-9)}$$

The FT of the partial derivative of a function (provided $g(y)$ approaches zero sufficiently rapidly at large y) is

$$\mathcal{F}\left(\frac{\partial}{\partial y} g(y), f\right) = -jfG(f) \ . \qquad \text{(A-10)}$$

Other notation conventions were included in the text of this report.

Appendix B.—Equation Solution Procedure

The Fourier transform (FT) shown in equation (22) of this report, is reproduced here in the equation below.

$$\partial_t \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 C_1(q',t)C_1(q-q',t) = -Kq_i^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 C_1(q',t)C_1(q-q',t) +$$

$$2K\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3[(q_i')(q_i - q_i')]C_1(q',t)C_1(q-q',t) \tag{B-1}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3[\partial_t + Kq_i^2 - 2K(q_i')(q_i - q_i')]C_1(q',t)C_1(q-q',t) = 0\,.$$

The solution to equation (B-1) is carried out in equation (B-2), as well as conversion back into the space domain.

To ensure that the integral in the second line of equation (B-1) is zero, let the integrand be zero. Temporarily indicating the vorticity product on the right as the function $P(q, q', t)$, the Laplace transform (LT) of the integrand may be calculated as

$$\begin{aligned}
\mathscr{L}\{P(q,q',t)\} &= \wp(q,q',s) = \int_0^{\infty} dt\, P(q,q',t)\exp(-st)\\
[\partial_t + K(q_i)^2 - 2K(q_i')(q_i - q_i')]P(q,q',t) &= 0\\
[s + K(q_i)^2 - 2K(q_i')(q_i - q_i')]\wp(q,q',s) &= P(q,q',0)\\
\wp(q,q',s) &= \frac{P(q,q',0)}{[s + K(q_i)^2 - 2K(q_i')(q_i - q_i')]}\\
P(q,q',t) &= P(q,q',0)\exp\{-K[(q_i)^2 - 2(q_i')(q_i - q_i')]t\}\,.
\end{aligned} \tag{B-2}$$

The first line in equation B-2 defines the LT. The second line is the integrand from equation (B-1). The third and fifth lines of the equation are standard LT [6]. The left-hand side of the last line of the equation, minus the right-hand side may now be substituted for the integrand of the last line of equation (B-1), as in

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 C_1(q',t)C_1(q-q',t) =$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 C_1(q',0)C_1(q-q',0)\exp\{-K[(q_i)^2 - 2(q_i')(q_i - q_i')]t\}\,. \tag{B-3}$$

The next step involves the IFT of equation (23). Addressing the left-hand side, the IFT integration will be applied, and the integral expression for the C's will be substituted, as in

$$
\begin{aligned}
\mathcal{F}^{-1}\{\mathrm{LHS}\} &= (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3\exp(-jq_ix_i)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3C_1(q',t)C_1(q-q',t)\\
&= (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3\exp(-jq_ix_i)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3\\
&\quad\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dx')^3\exp(jq_i'x_i')c_1(x',t)\\
&\quad\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dx'')^3\exp[j(q_i-q_i')x_i'']c_1(x'',t)\\
&= (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dx')^3\exp(jq_i'x_i')c_1(x',t)\\
&\quad\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dx'')^3\exp[-jq_i'x_i'']c_1(x'',t)\\
&\quad\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3\exp[jq_i(x_i''-x_i)]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dx')^3\exp(jq_i'x_i')c_1(x',t)\exp[-jq_i'x_i]c_1(x,t)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dx')^3c_1(x',t)c_1(x,t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3\exp[jq_i'(x_i'-x_i)]\\
&= (2\pi)^{-3}\,[c_1(x,t)]^2\,.
\end{aligned}
\tag{B-4}
$$

The square of the vorticity was defined to be the enstrophy in section 4

$$
\begin{aligned}
e(x,t) &= [c_j(x,t)]^2\\
\mathcal{F}\{e(x,t)\} = E(q,t) &= (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3C_j(q',t)C_j(q-q',t)\\
E(q,t) &= (2\pi)^{-3}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3P(q,q',t)\,.
\end{aligned}
\tag{B-5}
$$

The IFT of the last expression in equation (B-3) may now be written as

$$e_1(x,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3\exp(-jq_ix_i)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3C_1(q',0)C_1(q - q',0)$$

$$\exp\{-K[(q_i)^2 - 2(q_i')(q_i - q_i')]t\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3C_1(q',0)\exp[K(q_i')^2t] \qquad \text{(B-6)}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3C_1(q - q',0)\exp\{-K[(q_i)^2 - 2q_i'q_i]t\}\exp(-jq_ix_i).$$

Replacing $C_i(q,q',0)$ from equation (15) for all components produces the following equation:

$$e_1(x,t) = (\pi^3/4)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3q_1'q_3'\exp[-(q_i')^2/4]\exp[-K(q_i')^2t]\exp(-jq_i'x_i)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3(q_1 - q_1')(q_3 - q_3')\exp[-(q_i - q_i')^2/4]$$

$$\exp[-K(q_i - q_i')^2t]\exp[-j(q_i - q_i')x_i]$$

$$e_2(x,t) = (\pi^3/4)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3q_2'q_3'\exp[-(q_i')^2/4]\exp[-K(q_i')^2t]\exp(-jq_i'x_i)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3(q_2 - q_2')(q_3 - q_3')\exp[-(q_i - q_i')^2/4]$$

$$\exp[-K(q_i - q_i')^2t]\exp[-j(q_i - q_i')x_i] \qquad \text{(B-7)}$$

$$e_3(x,t) = (\pi^3/4)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3[(q_2')^2 + (q_2')^2]\exp[-(q_i')^2/4]$$

$$\exp[-K(q_i')^2t]\exp(-jq_i'x_i)$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq)^3[(q_1 - q_1')^2 + (q_2 - q_2')^2]\exp[-(q_i - q_i')^2/4]$$

$$\exp[-K(q_i - q_i')^2t]\exp[-j(q_i - q_i')x_i].$$

The right-most integrals are IFTs and produce functions that have no q' dependence. The notation has been simplified by setting $L = 1/4 + Kt$.

$$e_1(x,t) = (\pi^3/4)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 q_1' q_3' \exp[-(q_i')^2/4]\exp[-K(q_i')^2 t]\exp(-jq_i' x_i)$$
$$\left(\frac{-x_1 x_3 \exp[-x_i^2/(4L)]}{32L^{7/2}\pi^{3/2}}\right)$$
$$e_2(x,t) = (\pi^3/4)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3 q_2' q_3' \exp[-(q_i')^2/4]\exp[-K(q_i')^2 t]\exp(-jq_i' x_i)$$
$$\left(\frac{-x_2 x_3 \exp[-x_i^2/(4L)]}{32L^{7/2}\pi^{3/2}}\right) \tag{B-8}$$
$$e_3(x,t) = (\pi^3/4)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(dq')^3[(q_2')^2 + (q_2')^2]\exp[-(q_i')^2/4]$$
$$\exp[-K(q_i')^2 t]\exp(-jq_i' x_i)\left(\frac{(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)]}{32L^{7/2}\pi^{3/2}}\right).$$

The integrals in equation (B-8) are the same as the right-most integrals of equation (B-7), so that the result simply squares the ( ) brackets.

$$e(x,t) = \left(\frac{4\exp[-2x_i^2/(4L)]}{(4L)^7}\right)[(x_1x_3)^2 + (x_2x_3)^2 + (4L - x_1^2 - x_2^2)^2] \; . \tag{B-9}$$

Equation (B-9) is the formal result of this investigation, and was reported in section 4 as equation (24).

Appendix C.—Details of the Confirmation Process

The purpose of this appendix is to record the details confirming that the enstrophy expression, equation (24) of the main report, is a solution of the transport equation for instantaneous enstrophy, equation (27). In section 5, trial vorticity and trial velocity functions (denoted by bold letters **c** and **v**, respectively) were inferred from equation (24) and are reproduced here in equations (C-1) and (C-2).

$$\mathbf{c} = \left(\frac{2\exp[-x_i^2/(4L)]}{(4L)^{7/2}}\right)[\hat{x}_1 x_1 x_3 + \hat{x}_2 x_2 x_3 + \hat{x}_3(4L - x_1^2 - x_2^2)] \ . \tag{C-1}$$

$$\mathbf{v} = [1/(4L)^{5/2}](-\hat{x}_1 x_2 + \hat{x}_2 x_1)\exp[-x_i^2/(4L)] \ . \tag{C-2}$$

I showed in section 5 that the divergence of **c** and **v** are zero—a condition established early in this investigation. The confirmation process that began in section 5 identified a function Λ_C made up of the three terms of equation (27). These three terms were separately identified in equation (31), which is reproduced here as equation (C-3).

$$\begin{aligned} \Lambda_c &= \boldsymbol{\alpha}_c - \boldsymbol{\beta}_c - K\boldsymbol{\varphi}_c \\ \boldsymbol{\alpha}_c &= c_j \partial_t c_j \\ \boldsymbol{\beta}_c &= c_j s_j \\ \boldsymbol{\varphi}_c &= c_j (\partial_i)^2 c_j \ . \end{aligned} \tag{C-3}$$

Expansion of function $\boldsymbol{\alpha}_c$ is formulated in equation (C-4).

$$\begin{aligned} \boldsymbol{\alpha}_c &= \mathrm{c}_1\partial_t\mathrm{c}_1 + \mathrm{c}_2\partial_t\mathrm{c}_2 + \mathrm{c}_3\partial_t\mathrm{c}_3 \\ &= 2x_1x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\partial_t\{2x_1x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\} \\ &\quad + 2x_2x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\partial_t\{2x_2x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\} \\ &\quad + 2(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)](4L)^{-7/2} \\ &\quad \partial_t\{2(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)](4L)^{-7/2}\} \ . \end{aligned} \tag{C-4}$$

Taking the time derivatives and simplifying gives the following:

$$\begin{aligned} \boldsymbol{\alpha}_c = {} & [2x_1x_3]^2\exp[-x_i^2/(4L)](4L)^{-7/2}\{(4K)\exp[-x_i^2/(4L)](4L)^{-11/2} \\ & [x_i^2 - (7/2)(4L)]\} + [2x_2x_3]^2\exp[-x_i^2/(4L)](4L)^{-7/2}\{(4K) \\ & \exp[-x_i^2/(4L)](4L)^{-11/2}[x_i^2 - (7/2)(4L)]\} + 4(4L - x_1^2 - x_2^2) \\ & \exp[-x_i^2/(4L)](4L)^{-7/2}\{(4K)\exp[-x_i^2/(4L)](4L)^{-11/2} \\ & [(-7/2)(4L - x_1^2 - x_2^2)(4L) + x_i^2(4L - x_1^2 - x_2^2) + (4L)^2]\} \ . \end{aligned} \tag{C-5}$$

Even further simplification yields the following:

$$\begin{aligned}\alpha_c = {} & (8K)\exp[-2x_i^2/(4L)](4L)^{-9}\{x_3^2(x_1^2 + x_2^2)[2x_i^2 - 7(4L)] \\ & + 2x_i^2(4L - x_1^2 - x_2^2)^2 - 7(4L)(4L - x_1^2 - x_2^2)^2 \\ & + 2(4L)^2(4L - x_1^2 - x_2^2)\}.\end{aligned} \tag{C-6}$$

Expansion of function φ_c is shown in equation (C-7):

$$\begin{aligned}\varphi_c = {} & c_1\partial_i^2 c_1 + c_2\partial_i^2 c_2 + c_3\partial_i^2 c_3 \\ = {} & 2x_1x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\partial_i^2\{2x_1x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\} \\ & + 2x_2x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\partial_i^2\{2x_2x_3\exp[-x_i^2/(4L)](4L)^{-7/2}\} \\ & + 2(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)](4L)^{-7/2} \\ & \times \partial_i^2\{2(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)](4L)^{-7/2}\}.\end{aligned} \tag{C-7}$$

Breaking equation (C-7) into parts and performing the differentiations gives the following three equations.

$$\begin{aligned}\varphi_{c1} = {} & 4x_1x_3^2\exp[-x_i^2/(4L)](4L)^{-7}\partial_1^2\{x_1\exp[-x_i^2/(4L)]\} \\ & + 4x_2^2x_3^2\exp[-x_i^2/(4L)](4L)^{-7}\partial_1^2\{\exp[-x_i^2/(4L)]\} \\ & + 4(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)](4L)^{-7}\partial_1^2 \\ & \{(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)]\} \\ = {} & 8x_3^2x_1^2[2x_1^2/(4L) - 3]\exp[-2x_i^2/(4L)](4L)^{-8} \\ & + 8x_2^2x_3^2[2x_1^2/(4L) - 1]\exp[-2x_i^2/(4L)](4L)^{-8} \\ & + 8(4L - x_1^2 - x_2^2)[(8L - x_1^2 - x_2^2)(2x_1^2/(4L) - 1) + 2x_1^2] \\ & \exp[-2x_i^2/(4L)](4L)^{-8}.\end{aligned} \tag{C-8}$$

$$\begin{aligned}\varphi_{c2} = {} & 4[x_1x_3]^2\exp[-x_i^2/(4L)](4L)^{-7}\partial_2^2\{\exp[-x_i^2/(4L)]\} \\ & + 4x_2x_3^2\exp[-x_i^2/(4L)](4L)^{-7}\partial_2^2\{x_2\exp[-x_i^2/(4L)]\} \\ & + 4(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)](4L)^{-7}\partial_2^2 \\ & \{(4L - x_1^2 - x_2^2)\exp[-x_i^2/(4L)]\} \\ = {} & 8[x_1x_3]^2[2x_2^2/(4L) - 1]\exp[-2x_i^2/(4L)](4L)^{-8} \\ & + 8[x_2x_3]^2[2x_2^2/(4L) - 3]\exp[-2x_i^2/(4L)](4L)^{-8} \\ & + 8(4L - x_1^2 - x_2^2)[(8L - x_1^2 - x_2^2)(2x_2^2/(4L) - 1) + 2x_2^2] \\ & \exp[-2x_i^2/(4L)](4L)^{-8}.\end{aligned} \tag{C-9}$$

$$\begin{aligned}\varphi_{c3} &= 4x_1^2x_3\exp[-x_i^2/(4L)](4L)^{-7}\partial_3^2\{x_3\exp[-x_i^2/(4L)]\}\\ &+4x_2^2x_3\exp[-x_i^2/(4L)](4L)^{-7}\partial_3^2\{x_3\exp[-x_i^2/(4L)]\}\\ &+4(4L-x_1^2-x_2^2)\exp[-x_i^2/(4L)](4L)^{-7}\partial_3^2\\ &\{(4L-x_1^2-x_2^2)\exp[-x_i^2/(4L)]\}\\ &= 8x_1^2x_3^2[2x_3^2/(4L)-3]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8x_2^2x_3^2[2x_3^2/(4L)-3]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8[2x_3^2/(4L)-1](4L-x_1^2-x_2^2)^2\exp[-2x_i^2/(4L)](4L)^{-8}.\end{aligned}\tag{C-10}$$

Summing up the expressions in equations (C-8), (C-9), and (C-10) and simplifying the results achieves

$$\begin{aligned}\varphi_c &= 8x_3^2x_1^2[2x_1^2/(4L)-3]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8x_2^2x_3^2[2x_1^2/(4L)-1]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8(4L-x_1^2-x_2^2)[(8L-x_1^2-x_2^2)(2x_1^2/(4L)-1)+2x_1^2]\\ &\exp[-2x_i^2/(4L)](4L)^{-8}+8[x_1x_3]^2[2x_2^2/(4L)-1]\\ &\exp[-2x_i^2/(4L)](4L)^{-8}+8[x_2x_3]^2[2x_2^2/(4L)-3]\\ &\exp[-2x_i^2/(4L)](4L)^{-8}+8(4L-x_1^2-x_2^2)[(8L-x_1^2-x_2^2)\\ &(2x_2^2/(4L)-1)+2x_2^2]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8x_1^2x_3^2[2x_3^2/(4L)-3]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8x_2^2x_3^2[2x_3^2/(4L)-3]\exp[-2x_i^2/(4L)](4L)^{-8}\\ &+8[2x_3^2/(4L)-1](4L-x_1^2-x_2^2)^2\exp[-2x_i^2/(4L)](4L)^{-8}.\end{aligned}\tag{C-11}$$

Further simplification yields the following equations.

$$\begin{aligned}\varphi_c &= 8\exp[-2x_i^2/(4L)](4L)^{-9}\{[x_3x_1]^2[2x_1^2-3(4L)]+[x_2x_3]^2[2x_1^2-(4L)]\\ &+(4L-x_1^2-x_2^2)[(8L-x_1^2-x_2^2)(2x_1^2-(4L))+2(4L)x_1^2]\\ &+[x_1x_3]^2[2x_2^2-(4L)]+[x_2x_3]^2[2x_2^2-3(4L)]\\ &+(4L-x_1^2-x_2^2)[(8L-x_1^2-x_2^2)(2x_2^2-(4L))+2(4L)x_2^2]\\ &+[x_1x_3]^2[2x_3^2-3(4L)]+[x_2x_3]^2[2x_3^2-3(4L)]\\ &+[2x_3^2-4L](4L-x_1^2-x_2^2)^2\}.\end{aligned}\tag{C-12}$$

$$\begin{aligned}\varphi_c &= 8\exp[-2x_i^2/(4L)](4L)^{-9}\{x_3^2[x_1^2+x_2^2][2x_i^2-7(4L)]+2x_i^2(4L-x_1^2-x_2^2)^2\\ &-7(4L)(4L-x_1^2-x_2^2)^2+2(4L)^2(4L-x_1^2-x_2^2)\}.\end{aligned}\tag{C-13}$$

Comparing equation (C-6) with equation (C-13), we see that $\boldsymbol{\alpha}_\mathbf{c} = K\,\boldsymbol{\varphi}_\mathbf{c}$, so they will cancel each other out when they are entered into the Λ_C expression.

The second term from equation (C-3) is more conveniently written in vector notation as

$$\boldsymbol{\beta}_c = c \cdot [\nabla \times (\mathbf{v} \times \mathbf{c})] . \qquad \text{(C-14)}$$

First calculate the cross product of the velocity and the vorticity:

$$\mathbf{v} \times \mathbf{c} = -\exp[-2x_i^2/(4L)][(4L)^{-6}/2]\hat{x}_3 x_3 [x_1^2 + x_2^2] \cdot \qquad \text{(C-15)}$$

Next, find the curl of the expression in equation (C-15), which will be the function **s**, as in

$$\mathbf{s} = \exp[-2x_i^2/(4L)][(4L)^{-6}]x_3[4L - x_1^2 - x_2^2][-\hat{x}_1 x_2 + \hat{x}_2 x_1] \cdot \qquad \text{(C-16)}$$

The function $\boldsymbol{\beta}_\mathbf{c}$ will then be the dot product of **c** with the above, or

$$\begin{aligned}\boldsymbol{\beta}_c = {} & \left(\frac{2\exp[-3x_i^2/(4L)]}{(4L)^{19/2}}\right) x_3 [4L - x_1^2 - x_2^2] \\ & [\hat{x}_1 x_1 x_3 + \hat{x}_2 x_2 x_3 + \hat{x}_3 (4L - x_1^2 - x_2^2)] [-\hat{x}_1 x_2 + \hat{x}_2 x_1] \equiv 0 .\end{aligned} \qquad \text{(C-17)}$$

Therefore, the second term is identically zero.

Appendix D.—Analysis of the Enstrophy Equation Applied to the Navier-Stokes Equation

In this appendix, the inferred time-dependent velocity field will be applied to the Navier-Stokes (N-S) equation to determine if it is a solution. The form of N-S to be used is equation (2), which is reproduced here as equation (D-1).

$$\frac{\partial u}{\partial \tau} = -\left(\frac{1}{\rho}\right)\nabla p - (1/2)\nabla u^2 + u\times(\nabla\times u) + \left(\frac{\eta}{\rho}\right)\nabla^2 u + \left(\frac{\zeta}{\rho} + \frac{\eta}{3\rho}\right)\nabla(\nabla\cdot u) \quad . \tag{D-1}$$

The trial vorticity and velocity field equations are reproduced here from equation (32).

$$d(y,\tau) = \left(\frac{2\Omega_0 a^5 \exp[-y_i^2/(a^2 + 4\nu\tau)]}{(a^2 + 4\nu\tau)^{7/2}}\right)[\hat{y}_1 y_1 y_3 + \hat{y}_2 y_2 y_3 + \hat{y}_3(a^2 + 4\nu\tau - y_1^2 - y_2^2)]$$

$$u(y,\tau) = \left(\frac{\Omega_0 a^5 \exp[-y_i^2/(a^2 + 4\nu\tau)]}{(a^2 + 4\nu\tau)^{5/2}}\right)[-\hat{y}_1 y_2 + \hat{y}_2 y_1] \, . \tag{D-2}$$

An expression for the pressure will be required; that expression will be developed after the form of equation (34) and shall be shown as equation (D-3).

$$r'dp = \rho(u^2/r')r'dr' \qquad r' = \sqrt{y'^2_1 + y'^2_2}$$

$$p(\infty,\tau) - p(y,\tau) = 2\int_{\sqrt{y_1^2+y_2^2}}^{\infty} dr' w(r', y_3, \tau)/r'$$

$$= \left(\frac{\Omega_0^2 a^{10}\rho}{4(a^2 + 4\nu\tau)^5}\right)\exp[-2y_i^2/(a^2 + 4\nu\tau)] \tag{D-3}$$

$$p(y,\tau) = p(\infty,\tau) - \left(\frac{\Omega_0^2 a^{10}\rho}{4(a^2 + 4\nu\tau)^5}\right)\exp[-2y_i^2/(a^2 + 4\nu\tau)] \quad .$$

To carry out the confirmation process, trial expressions will be identified by bold symbols. Λ_u will represent equation (D-1) with all the terms moved to the left-hand side (except for the last term, which is zero because the divergence of the velocity is zero). If Λ_u is zero when the velocity expression of equation (D-2) is substituted, equation (D-2) is a solution to equation (D-1). Two trial expressions will also be calculated: Λ_1 and Λ_2. These functions represent the differently orientationed components of equation (D-1). If Λ_1 and Λ_2 are simultaneously zero, Λ_u will be zero. These identifications are made in the equation (D-4).

$$\Lambda_u = \Lambda_1 + \Lambda_2 \quad \Lambda_1 = \boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1 + \boldsymbol{\varphi}_1 \quad \Lambda_2 = \boldsymbol{\alpha}_2 + \boldsymbol{\beta}_2$$

$$\boldsymbol{\alpha}_1 = \left(\frac{1}{\rho}\right)\nabla p = \left(\frac{\Omega_0^2 a^{10}}{\left(a^2 + 4\nu\tau\right)^6}\right)[\hat{y}_i y_i]\exp\left[-2y_i^2/\left(a^2 + 4\nu\tau\right)\right]$$

$$\boldsymbol{\beta}_1 = (1/2)\nabla \mathbf{u}^2 = \left\{\frac{\Omega_0^2 a^{10}}{2\left(a^2 + 4\nu\tau\right)^5}\right\}\nabla\left\{\left[y_1^2 + y_1^2\right]\exp\left[-2y_i^2/\left(a^2 + 4\nu\tau\right)\right]\right\}$$

$$\boldsymbol{\varphi}_1 = \mathbf{u} \times \mathbf{d} = -\left\{\frac{2\Omega_0^2 a^{10}}{\left(a^2 + 4\nu\tau\right)^6}\right\}\exp\left[-2y_i^2/\left(a^2 + 4\nu\tau\right)\right]$$

$$\left\{\left[-\hat{y}_1 y_2 + \hat{y}_2 y_1\right] \times \left[\hat{y}_1 y_1 y_3 + \hat{y}_2 y_2 y_3 + \hat{y}_3\left(a^2 + 4\nu\tau - y_1^2 - y_2^2\right)\right]\right\} \tag{D-4}$$

$$\boldsymbol{\alpha}_2 = \partial_\tau \mathbf{u} = \partial_\tau\left\{\frac{\Omega_0 a^5}{\left(a^2 + 4\nu\tau\right)^{5/2}}\right\}\left\{\left[-\hat{y}_1 y_2 + \hat{y}_2 y_1\right]\exp\left[-y_i^2/\left(a^2 + 4\nu\tau\right)\right]\right\}$$

$$\boldsymbol{\beta}_2 = \nu\nabla^2\mathbf{u} = -\nu\left\{\frac{\Omega_0 a^5}{\left(a^2 + 4\nu\tau\right)^{5/2}}\right\}\nabla^2\left\{\left[-\hat{y}_1 y_2 + \hat{y}_2 y_1\right]\exp\left[-y_i^2/\left(a^2 + 4\nu\tau\right)\right]\right\}.$$

The gradient operation has already been accomplished for $\boldsymbol{\alpha}_1$, and no further simplification is needed. Next, simplify the $\boldsymbol{\beta}_1$ function:

$$\boldsymbol{\beta}_1 = \left(\frac{\Omega_0^2 a^{10}}{(a^2 + 4\nu\tau)^6}\right)\{(a^2 + 4\nu\tau)[\hat{y}_1 y_1 + \hat{y}_2 y_2] - 2[y_1^2 + y_2^2]$$
$$[\hat{y}_1 y_1 + \hat{y}_2 y_2 + \hat{y}_3 \ y_3]\}\exp[-2y_i^2/(a^2 + 4\nu\tau)] \quad . \tag{D-5}$$

The simplify the $\boldsymbol{\varphi}_1$ function:

$$\boldsymbol{\varphi}_1 = -\left(\frac{2\Omega_0^2 a^{10}}{(a^2 + 4\nu\tau)^6}\right)\exp[-2y_i^2/(a^2 + 4\nu\tau)][\hat{y}_1 y_1(a^2 + 4\nu\tau - y_1^2 - y_2^2)$$
$$+\hat{y}_2 y_2(a^2 + 4\nu\tau - y_1^2 - y_2^2) - \hat{y}_3 y_3(y_1^2 + y_2^2)] \quad . \tag{D-6}$$

Find the sum of $\boldsymbol{\alpha}_1$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\varphi}_1$:

$$\Lambda_1 = \boldsymbol{\alpha}_1 + \boldsymbol{\beta}_1 + \boldsymbol{\varphi}_1$$
$$= \left(\frac{\Omega_0^2 a^{10}\exp[-2y_i^2/(a^2 + 4\nu\tau)]}{(a^2 + 4\nu\tau)^6}\right)\{[\hat{y}_i y_i] - (a^2 + 4\nu\tau)[\hat{y}_1 y_1 + \hat{y}_2 y_2]\} \quad . \tag{D-7}$$

In doing so, we find that Λ_1 is not zero. Proceed by simplifying the Λ_2 function by first expanding $\boldsymbol{\alpha}_2$:

$$\boldsymbol{\alpha}_2 = \left(\frac{\Omega_0 a^5}{(a^2 + 4\nu\tau)^{9/2}}\right)[-\hat{y}_1 y_2 + \hat{y}_2 y_1]\exp[-y_i^2/(a^2 + 4\nu\tau)]$$
$$(4\nu)[y_i^2 - (5/2)(\mathrm{a}^2 + 4\nu\tau)] \quad . \tag{D-8}$$

Finally, expand $\boldsymbol{\beta}_2$:

$$\boldsymbol{\beta}_2 = -4\nu\left(\frac{\Omega_0 a^5}{(a^2 + 4\nu\tau)^{9/2}}\right)[-\hat{y}_1 y_2 + \hat{y}_2 y_1]\exp[-y_i^2/(a^2 + 4\nu\tau)] \\ [y_i^2 - 2(a^2 + 4\nu\tau)] \quad . \tag{D-9}$$

Comparing equation (D-8) with equation (D-9), we see that Λ_2 is not zero, although the difference has the time dependence of the vorticity, but is in the opposite direction of the velocity.

Since Λ_1 and Λ_2 are each not zero and have different dependence in the exponential factor, their sums Λ cannot be zero, and hence equation (D-2) is not a solution to equation (D-1).

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13. ABSTRACT *(Maximum 200 words)* This investigation was undertaken as a part of a larger effort to determine the implications of using coherent turbulent structures or turbules to calculate acoustic scattering from turbulent media, a process that has been suggested as a possible non-line-of-sight means of detecting enemy assets on the battlefield. I present an analytical solution to a correlate of the incompressible Navier-Stokes equation. The correlate equation is for the enstrophy, the square of the vorticity. The solution is an expression for the time history of the enstrophy for a particular choice of the initial velocity distribution.			
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